

Theoretical Physics Institute University of Minnesota

TPI-MINN-96/07-T
UMN-TH-1428-96
hep-ph/9606281

Little Miracles of Supersymmetric Evolution of Gauge Couplings

Talk * at the 4th International Workshop on
Supersymmetry and Unification of Fundamental Interactions
SUSY-96 (May 29 – June 1)
University of Maryland, College Park, MD 20742, USA

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Abstract

The invention of supersymmetry, almost exactly 25 years ago [1], changed the face of high-energy physics. The idea that the observed low-energy gauge groups appear due to the process of spontaneous breaking of a single unifying group G is also quite popular. The synthesis of these two elements results in supersymmetric grand unification. I present (perturbatively) exact results regarding the supersymmetric evolution of the gauge couplings from the scale of their unification to lower scales. In particular, it is shown how the heavy mass thresholds can be properly taken into account to all orders.

*Extended version

When I was kindly invited by Prof. Mohapatra, almost a year ago, to give this talk, it was supposed that the main subject would be the α_s crisis [2]. Since then the situation with low versus high-energy determinations of $\alpha_s(M_Z)$ was reviewed several times [3], and nothing exciting happened during this year. The irreconcilable contradiction between the values of $\alpha_s(M_Z)$ from the low-energy measurements and the high energy determinations is not going to disappear – this fact gradually becomes obvious to everybody. Perhaps, it is worth mentioning a very recent record-breaking analysis of deep inelastic scattering [4] which achieves three-loop accuracy, for the first time ever in this problem. The analysis confirms, with better stability, the earlier conclusion, that $\alpha_s(M_Z)$ is equal to 0.110 with the uncertainty close to 0.005. According to Ref. [4], this is a conservative estimate of the error bars. If we compare this result with the values of α_s extracted from the measurements at the Z peak we observe a discrepancy of 2.5 to 3 standard deviations. Thus, the question is not whether or not new physics is already with us but, rather, what kind of new physics manifests itself in the existing data. Quite a few talks at this Workshop were devoted to various new physics scenarios, expressing various degrees of confidence in the scenarios under consideration [5]. I have nothing to add in this respect. In the absence of fresh ideas concerning the α_s crisis, I decided to discuss today another topic which, on one hand, is rather closely related to the α_s problem and, on the other hand, did not receive due attention previously – a very peculiar pattern of the supersymmetric evolution of the gauge couplings.

1 Outlining the Problem

The bold idea of grand unification, after two decades of its existence [6], has become an indispensable part of phenomenology. It has obviously attractive features, e.g. the fact that it naturally explains the electric charge quantization. Its most unpleasant feature, to my mind, is the prediction of a great desert above the supersymmetry scale (100 GeV range) up to the grand unification scale (10^{16} GeV range), an energy interval spanning fourteen orders of magnitude or so, where nothing happens. If the great desert does indeed exist, high energy physics will practically cease to exist as the experimental science we know today, after the discovery of the superpartners. I would not like to believe in such a frightening future, and, yet, I am going to discuss the supersymmetric grand unification, for two reasons. First, I will emphasize some general aspects of supersymmetric evolution of the gauge couplings which may well be of importance in other problems, not necessarily related to geography of the great desert. Treating the gauge coupling evolution in a standard way [7], one overlooks elegant properties which are specific to supersymmetry. My task is to reveal these elegant properties. Second, supersymmetric grand unification *per se* is an intellectual game occupying the minds of many, probably, because it predicts a relation between the low-energy gauge couplings which is surprisingly close to the one observed experimentally. It seems worth joining the club of unifiers merely to

understand whether the experimentally observed relation is actually reproduced in the most naive and straightforward version of supersymmetric grand unification.

In the first part of my talk I will focus on the generalities of the supersymmetric evolution of the gauge couplings. Assume that there is a (simple) gauge group G which is spontaneously broken down to several subgroups. Far above the symmetry breaking scale we start from a single gauge coupling. Then it splits, and far below the breaking scale we deal with a whole set of independent couplings which, however, remember their common origin. The common origin imposes certain constraints.

There is nothing new in this formulation, of course. Getting observable consequences requires evolving the constants from the unification scale down to a lower scale – this fact was recognized almost immediately [8] after the emergence of the grand unified theories (GUT). Huge amounts of literature are devoted to this issue. Let me mention early works on supersymmetric grand unification [9], and more recent famous works [10] which essentially buried (at least, psychologically) non-supersymmetric unification. All analyses of the supersymmetric evolution of which I am aware borrow the old technique used in the non-supersymmetric case [8]. A new element which I would like to promote is the full exploitation of very specific features of the supersymmetric β functions [11]. Among other virtues, this new approach allows one to exactly solve (*to all orders*) the problem of the supersymmetric heavy thresholds.

Supersymmetric grand unification, in the context of the realistic model-building, is briefly discussed in the second part. Here some simple numerical estimates are presented explaining the impact of various effects on $\alpha_s(M_Z)$.

2 Full Beta Function from One Loop of Perturbation Theory

First, I would like to explain how the full multiloop (perturbative) β function can be obtained from a trivial *one-loop* calculation. The derivation presented below was reported over a decade ago [12]. Since this paper is hardly known to anybody it is instructive to reiterate the main points, the more so that we will need them later on, in discussing the threshold effects.

Let us consider a generic non-Abelian gauge theory with the matter sector which includes (i) one chiral superfield in the adjoint representation of the gauge group (ii) additional matter superfields in arbitrary representations. For simplicity I will keep in mind that the gauge group G is $SU(N)$ and assume that the additional matter is such that a mass term is possible for every superfield. Both assumptions can be lifted; moreover, the final expression we will obtain shortly is absolutely general.

In more or less standard (and somewhat symbolic) notation the action has the form

$$\mathcal{S}_0 = \frac{1}{2g_0^2} \text{Tr} \int d^2\theta d^4x W^2 + \frac{1}{2g_0^2} \text{Tr} \int d^2\theta d^2\bar{\theta} d^4x \bar{\Phi} e^V \Phi$$

$$+ \frac{1}{4} \int d^2\theta d^2\bar{\theta} \sum_f S_f^\dagger e^V S_f + \left(\frac{1}{4} m_f^0 \int d^2\theta (S_f)^2 + \text{h. c.} \right) \quad (1)$$

where g_0 is the bare coupling constant, Φ is the superfield in the adjoint representation,

$$\Phi \equiv \sum_{a=1}^{N^2-1} \Phi^a T^a,$$

S_f is the set of all other superfields, f is the “flavor” index, and m_f^0 is the bare mass term. Finally, T^a are $N \times N$ traceless matrices, generators of $SU(N)$. Let us note that the normalizations of the matter fields are different – Φ includes the coupling constant while S_f ’s do not. As will become clear shortly, this is convenient. The bare Z factors of all fields S_f are set equal to unity. The lowest component of the superfield Φ will be denoted by φ .

One may add all conceivable (super) Yukawa interactions between S_f ; this does not change our argument. Perhaps, it is worth noting that in the absence of the additional S_f matter fields we actually deal with the extended $N = 2$ supersymmetry.

The model described by the action (1) possesses D -flat directions – a system of degenerate classical vacua in the space of Φ fields, the so called valleys; the potential energy along the bottom of the valley vanishes. The valleys are parametrized by $N - 1$ chiral invariants, $\text{Tr } \Phi^2, \text{Tr } \Phi^3, \dots, \text{Tr } \Phi^{N-1}$. Alternatively, one can say that for any diagonal matrix Φ the D terms vanish. For instance, for $SU(3)$ choosing $\varphi^3 = a$ a complex constant and $\varphi^8 = b$ a complex constant and putting all other φ ’s to zero we get a point belonging to the bottom of the valley.

There is only one point, namely the origin ($\varphi^a = 0$ for all a), where the full original gauge symmetry is unbroken. Any other point from the bottom of the valley corresponds to the spontaneous breaking of the symmetry. (This statement is valid, generally speaking, only in perturbation theory. Remember, however, that I am discussing only perturbation theory). Generically, $SU(N)$ is broken down to $U(1)^{N-1}$. As a result, we have $N - 1$ massless “photons” with their superpartners, and $N^2 - N$ massive “ W bosons”, with their superpartners. These “ W bosons” are analogs of the superheavy X and Y bosons of the standard theory of grand unification, sometimes called elephants. (Quite naturally – elephants live in the areas at the far side of the desert! All gauge bosons of the group G which acquire masses through the expectation value of the scalar field belonging to the adjoint representation of G will be referred to below as elephants.) Their masses need not be equal to each other. For example, for $SU(3)$

$$M_{1,2}^2 = |\varphi^3|^2, \\ M_{4,5}^2 = \frac{1}{4} |\sqrt{3}\varphi^8 + \varphi^3|^2, \quad M_{6,7}^2 = \frac{1}{4} |\sqrt{3}\varphi^8 - \varphi^3|^2. \quad (2)$$

Now, we start our evolution from the normalization point μ equal to the ultra-violet cut off M_0 , where the action coincides with the bare one, Eq. (1), and then

descend down from M_0 to some low normalization point. M_0 is assumed to be much larger than any other scale in the problem at hand. As we cross the mass thresholds we integrate out the corresponding massive fields, one after another. In this way we integrate out the elephants and all additional matter fields S_f . In the low-energy limit the only surviving fields are the $N - 1$ massless photons (and their superpartners), which are mutually neutral with respect to each other and do not interact. Since the residual gauge symmetry is $U(1)^{N-1}$ and there is no charged matter left below the last threshold, the gauge couplings do not run below the last threshold. There are $N - 1$ low-energy gauge couplings corresponding to $N - 1$ unbroken $U(1)$ subgroups. These charges obtained after the completion of the evolution will be referred to as low-energy (frozen) charges α . They could be measured in a *gedanken* experiment, say, through the Coulomb interaction of heavy probe charged particles at large distances.

We are free to choose any low-energy (frozen) charge and consider it as a function of the ultraviolet cut off, the bare coupling constant $\alpha_0 \equiv g_0^2/4\pi$ and other parameters of the theory. Varying M_0 and α_0 in a concerted way, ensuring that the low-energy quantities stay fixed, we find the M_0 dependence of α_0 which is equivalent to the knowledge of the β function. For definiteness, one may concentrate on the charge corresponding to the “third photon”, A_μ^3 , although this specific choice is of no importance for our purpose in this part.

To connect α and α_0 one needs an explicit formula for the low-energy effective action,

$$\mathcal{S} = \frac{1}{4} \left(\frac{1}{g^2} \right)_{ij} \int d^2\theta d^4x W^i W^j + \frac{Z_{ij}}{4g_0^2} \int d^2\theta d^2\bar{\theta} d^4x \bar{\Phi}^i \Phi^j. \quad (3)$$

This action includes only photons and the neutral adjoint matter. Correspondingly, the indices i, j run over the Cartan subalgebra (3 and 8 in the $SU(3)$ example; in general they take $\text{rank}(G)$ different values), and Z_{ij} is the set of renormalization Z factors of the neutral adjoint matter fields evolved down “to the very end”. Note that the kinetic term of the low-energy photons need not be diagonal in i and j .

Limiting ourselves to the third photon gauge coupling ($i = j = 3$) one can trivially write the expression for $1/g^2$ at one loop,

$$\frac{1}{g^2} = \frac{1}{g_0^2} - \frac{2N}{8\pi^2} \ln \frac{M_0}{\tilde{\varphi}_0} + \sum_f \frac{T(R_f)}{8\pi^2} \ln \frac{M_0}{m_f^0}. \quad (4)$$

Here $T(R_f)$ is (one half of) the Dynkin index,

$$\text{Tr}(T^a T^b) = T(R) \delta^{ab},$$

T^a are the group generators in the representation corresponding to the superfield S_f . For the adjoint representation of $SU(N)$ the corresponding index $T(\text{adjoint}) = T(G) = N$. This fact is actually used in Eq. (4); $2N$ in front of the first logarithm is actually $2T(G)$. For the fundamental representation of $SU(N)$ the index $T(\text{fund}) =$

1/2 for each chiral superfield. It is worth reminding that one flavor requires two chiral superfields in this case, so that effectively $T = 1$ for each flavor. Finally, $\tilde{\varphi}$ is a homogeneous function of the moduli whose concrete form depends on the group under consideration. In the $SU(3)$ case for the third photon

$$\tilde{\varphi} = (\varphi^3)^{2/3} \left[\frac{1}{4} (\varphi^3)^2 - \frac{3}{4} (\varphi^8)^2 \right]^{1/6}.$$

This rather clumsy expression is nothing but a combination of masses (2). It is important to note that $\tilde{\varphi}$ depends only on φ , not on φ^\dagger , and the only singular points (i.e. those where $\tilde{\varphi} = 0$) are

$$\varphi^3 = 0 \text{ or } \varphi^3 = \pm \sqrt{3} \varphi^8. \quad (5)$$

At these points the pattern of the symmetry breaking is not generic; instead of $SU(3) \rightarrow U(1)^2$ we have $SU(3) \rightarrow U(1) \times SU(2)$, there are massless non-neutral gauge bosons, and the notion of the low-energy frozen action becomes ill-defined. We will stay away from the singular points.

Now we come to a miracle. Although the low-energy coupling constant (4) was obtained at one loop this expression is (perturbatively) exact to all orders – the gauge part of the effective action (3) receives no correction at the level of two loops, three loops, and so on – at every finite order. This non-renormalization theorem stems from the fact that *if the theory is fully regularized in the infrared domain* there are no holomorphic anomalies [13], and the coefficient in front of W^2 in the effective action must be an analytic function of g_0^{-2} , m_f^0 and the moduli fields [14, 15, 17]. In the context of the string theory a similar observation was made in Ref. [18]. A more technical proof relying on specific properties of the background field technique for supergraphs was given in [15] (see also [16]) where the reader will also find a comprehensive list of earlier works in this direction. Note that the vacuum expectation values of the fields φ in all expressions above are those of the bare fields – the assertion of the holomorphic dependence refers to the *bare* parameters. Had the coefficient in front of W^2 been expressed in terms of the renormalized (low-energy) parameters, the holomorphicity would be lost. To be consistent, I should have marked the field $\tilde{\varphi}$ by the subscript “0”; I will do this occasionally, but not in every expression. The proliferation of indices makes them unreadable, so sometimes we will just keep this subscript in mind.

It is very easy to see why higher corrections can not appear. Say, the two-loop term in the coefficient of W^2 , if it existed, should be proportional to $1/(\text{Re } g_0^{-2})$, since the imaginary part of g_0^{-2} (the θ term) can not show up in the perturbative expansions. In which case there is no way to maintain the holomorphic dependence.

Let us pause here and reiterate. The relation between the physical gauge coupling and the bare one presented above is (perturbatively) exact. The left-hand side is required to be independent of the ultraviolet cut off. This requirement determines the M_0 dependence of g_0^{-2} and, hence, the full β function for the gauge coupling.

The β function is defined as

$$\beta(\alpha_0) = d\alpha_0/d \ln M_0.$$

Naively, one might conclude that the one-loop expression for the physical charge (4) yields the one-loop β function. Everybody knows that this conclusion is wrong, of course. So, there should be a subtlety. The question is “where?”.

Within our formulation the change of g_0^{-2} under the variation of M_0 should be adjusted in such a way as to keep *all* physical low-energy parameters fixed, not only g^{-2} . In particular, the physical (renormalized) masses of the matter fields and of the elephants must stay fixed; $\tilde{\varphi}$ and m_f differ from $\tilde{\varphi}_0$ and m_f^0 , however, by the corresponding Z factors, which, in turn, do depend on M_0 . Therefore, differentiating the right-hand side of Eq. (4) over $\ln M_0$ it is necessary to differentiate $\tilde{\varphi}_0$ and m_f^0 as well.

It is convenient to rewrite Eq. (4) as follows

$$\begin{aligned} \frac{1}{g^2} = & \left(\frac{1}{g_0^2} - \frac{2N}{8\pi^2} \ln \frac{M_0}{\tilde{\varphi}} + \sum_f \frac{T(R_f)}{8\pi^2} \ln \frac{M_0}{m_f} \right) \\ & + \left(\frac{2N}{8\pi^2} \ln \frac{\tilde{\varphi}_0}{\tilde{\varphi}} + \sum_f \frac{T(R_f)}{8\pi^2} \ln \frac{m_f}{m_f^0} \right). \end{aligned} \quad (6)$$

The sum of the first and the second brackets is holomorphic, each of them separately is not. The non-holomorphicity creeps in at this stage because the Z factors connecting, say, m_f and m_f^0 are not holomorphic. The second line in Eq. (6) reduces to

$$- \frac{1}{4\pi} \left(\frac{N}{2\pi} \ln \frac{Z\alpha}{\alpha_0} + \sum_f \frac{T(R_f)}{2\pi} \ln Z_f \right). \quad (7)$$

In the absence of the additional matter fields supersymmetry extends to $N = 2$, and the Z factor of the adjoint matter is $Z = \alpha_0/\alpha$. In other words, $\tilde{\varphi}$ is not renormalized in this case, and the β function is obviously one-loop.

In deriving Eq. (7) I used the fact that the mass term of the additional matter, being the F term, is not renormalized [19], and the entire renormalization of the mass of the f -th flavor comes from the corresponding Z_f factor. With the additional matter fields included, the $N = 2$ supersymmetry degrades to $N = 1$, and Eq. (7) generates the second and all higher-order terms in the β function. Indeed, differentiating α_0 over $\ln M_0$ it is trivial to get

$$\begin{aligned} 0 = & -\frac{1}{\alpha_0^2} \beta(\alpha_0) - \frac{2N}{2\pi} + \sum_f \frac{T(R_f)}{2\pi} + \\ & \frac{N}{2\pi} \frac{1}{\alpha_0} \beta \alpha_0 - \left(\frac{N}{2\pi} \gamma + \sum_f \frac{T(R_f)}{2\pi} \gamma_f \right) \end{aligned} \quad (8)$$

where γ is the anomalous dimension of the adjoint matter, γ_f is that of S_f ,

$$\gamma = d \ln Z / d \ln M_0$$

and for simplicity it is assumed that there is no flavor mixing in Z_f 's due to (possible) Yukawa interactions, which may be present, in principle, although I have not indicated them explicitly in Eq. (1). Needless to say that this assumption is not crucial. The first and second lines in Eq. (8) come from the first and second lines in Eq. (6), respectively. Combining various pieces together we arrive at

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi} \left(1 - \frac{N\alpha}{2\pi}\right)^{-1} \left[2N - \sum_f T(R_f) + N\gamma + \sum_f T(R_f)\gamma_f\right]. \quad (9)$$

This is nothing but a particular case (occurring when one of the matter fields is in the adjoint representation) of the general NSVZ β function [11]

$$\beta(\alpha) = -\frac{\alpha^2}{2\pi} \left(1 - \frac{T(G)\alpha}{2\pi}\right)^{-1} \left[3T(G) - \sum_i T(R_i)(1 - \gamma_i)\right] \quad (10)$$

where in the expression above the sum on the right-hand side runs over *all* matter fields [20].

3 Where have the higher orders gone?

Now, that I've explained how the multiloop β function appears, I am going to tell you that essentially it is never needed *per se*. Our task is calculating the gauge coupling; this calculation requires integration of the β function. But if we know the full expression for the gauge couplings beforehand why bother about getting the β function and then integrating it back to find the couplings?

If the evolution of the gauge couplings is complete and they are frozen, at the very “bottom”, one can represent the result as a pure one-loop expression provided the corresponding formulae are written in terms of the *bare*, not physical, threshold scales. Thus, if the bare values of the thresholds are given (as would be the case if a grand unified theory emerges, say, as a limit of string theory) we could just forget about the second and all higher loops altogether.

Let me elucidate what I mean by this rather paradoxical statement. As an example, consider the $SU(3)$ model described above, with one extra flavor in the fundamental representation (i.e. one chiral triplet superfield and one chiral antitriplet). Since our task here is purely illustrative it is convenient to assume that the symmetry breaking $SU(3) \rightarrow U(1)^2$ takes place in two stages: first at a high scale $SU(3) \rightarrow U(1) \times SU(2)$ and then, at a somewhat lower scale, the remaining $SU(2)$ is broken to $U(1)$. In other words, instead of dealing with the general situation (which can be addressed, of course) we will assume that $|\varphi^8| \gg |\varphi^3|$. Among other simplifications this will allow us to neglect the off-diagonal term

$$F_{\mu\nu}^{(3)} F_{\mu\nu}^{(8)}$$

in the low-energy effective action. Indeed, the coefficient in front of this term is

$$\frac{3}{32\sqrt{3}\pi^2} \left(\ln \frac{\varphi^8 - (\varphi^3/\sqrt{3})}{\varphi^8 + (\varphi^3/\sqrt{3})} \right);$$

in the limit at hand and is suppressed as $|\varphi^3/\varphi^8|$.

Under the above pattern of symmetry breaking the elephants split into two groups – four very heavy elephants, with mass squared equal to $3|\varphi^8|^2/4$ emerging at the first stage and two not so heavy elephants with mass squared $|\varphi^3|^2$ emerging at the second stage. There are four mass scales in the problem: the ultraviolet cut off M_0 , the elephant masses and the (bare) matter mass m_0 , with the following hierarchy

$$m_0 \ll |\varphi^3| \ll |\varphi^8| \ll M_0.$$

As we descend from M_0 , we pass the first elephant threshold where the gauge group $SU(3)$ is broken down to $U(1) \times SU(2)$. The matter triplet becomes an $SU(2)$ doublet plus a singlet coupled only to the eighth photon. Then, below the second elephant threshold, only the Abelian subgroup survives; we have two photons and three matter fields. The charges of these three matter fields with respect to these two photons are

$$\left(\frac{1}{2}, -\frac{1}{2}, 0 \right) \text{ for the third photon,}$$

and

$$\left(\frac{1}{2\sqrt{3}}, \frac{1}{2\sqrt{3}}, -\frac{1}{\sqrt{3}} \right) \text{ for the eighth photon.} \quad (11)$$

The gauge couplings α_3 and α_8 continue to evolve due to the contribution of the matter fields.

When we cross, in our descent, the first elephant threshold we get two separate β functions governing the evolution of α_8 and the $SU(2)$ gauge coupling. Then, at the second elephant threshold the $SU(2)$ gauge coupling becomes α_3 . Below the matter threshold the evolution of the gauge coupling constants stops, and we arrive to what we call frozen couplings.

The final answer for the low-energy (frozen) coupling constants is known exactly, as was demonstrated above. The all-order result for α_3 was given in Eq. (4). For the sake of convenience we reproduce it here again, along with the answer for α_8 ,

$$\begin{aligned} \frac{1}{\alpha_3} &= \frac{1}{\alpha_0} - \frac{6}{2\pi} \ln \frac{M_0}{(\varphi_0^3)^{2/3} (\sqrt{3}\varphi_0^8/2)^{1/3}} + \frac{1}{2\pi} \ln \frac{M_0}{m_f^0}, \\ \frac{1}{\alpha_8} &= \frac{1}{\alpha_0} - \frac{6}{2\pi} \ln \frac{M_0}{(\sqrt{3}\varphi_0^8/2)} + \frac{1}{2\pi} \ln \frac{M_0}{m_f^0}. \end{aligned} \quad (12)$$

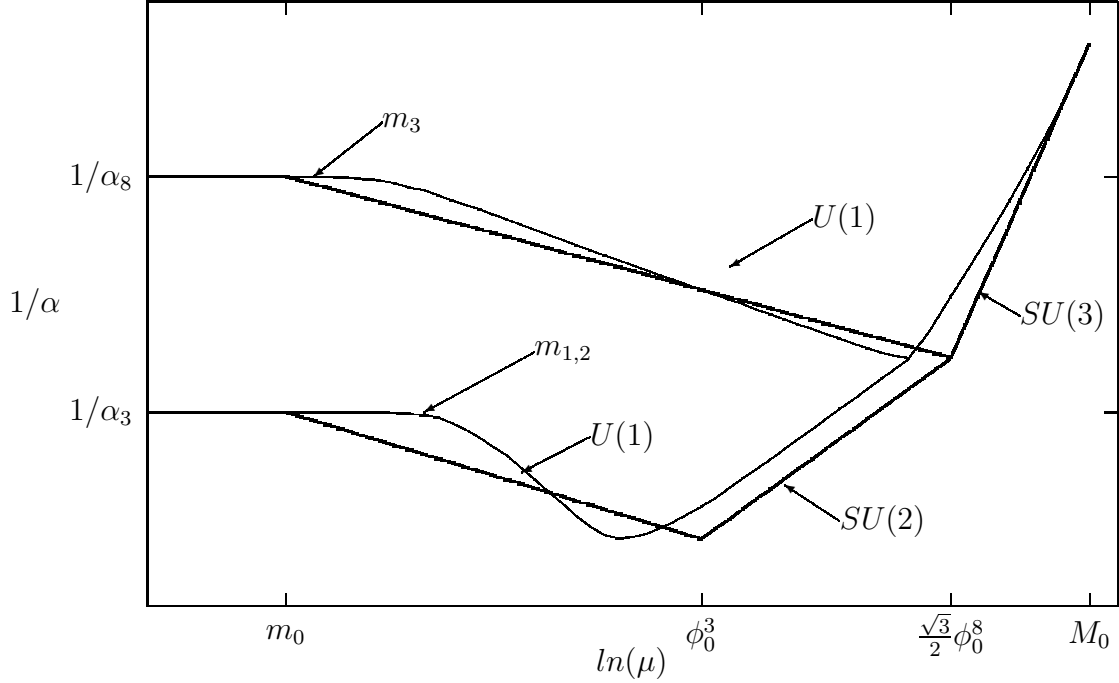


Figure 1: Evolution of the gauge couplings: from unification to freezing. The straight lines represent the one-loop evolution with the fake (bare) thresholds. The actual evolution is given by the smooth curves.

In the difference of the constants nearly everything drops out,

$$\frac{1}{\alpha_8} - \frac{1}{\alpha_3} = \frac{4}{2\pi} \ln \left(\frac{\sqrt{3}\varphi_0^8}{2\varphi_0^3} \right). \quad (13)$$

I emphasize that it is the ratio of the fields' bare expectation values that enters here. This result looks remarkably simple! One could superficially interpret it as follows. At the first (heaviest) elephant threshold, $\sqrt{3}\varphi_0^8/2$, the couplings $1/\alpha$ are unified. They diverge immediately, running linearly (in the log scale), according to the *one-loop* formula, all the way down. The slope of the linear running changes abruptly at the elephant thresholds. From 5 above the heaviest elephant threshold it changes to 3 for the $SU(2)$ coupling and -1 for the eighth photon. Below the second elephant threshold, φ_0^3 , where we deal with the $U(1)$ couplings, they run in parallel, in accordance with the charge assignment (11), until both couplings reach one and the same threshold at m_0 , where they freeze at once (Fig. 1). This simple interpretation is operationally correct, but this is deceptive correctness, of course.

After briefly reflecting one gets a second thought. First of all, it is absolutely certain that the β functions are multiloop. Therefore, the running law is non-linear, and this non-linearity is different for $1/\alpha_3$ and $1/\alpha_8$, in particular, in the interval between the two elephant thresholds since the Z factors of the $SU(2)$ doublet and

singlet in this domain are definitely different. Second, for the same reason the matter threshold can not be at one and the same place for the third and the eighth photons – although we start from one and the same parameter m_0 the physical masses of the three fermions we get after $SU(3)$ is broken down to $U(1)^2$ are different. Therefore, the constants must freeze out at different scales. Finally, $\sqrt{3}\varphi_0^8/2$ and φ_0^3 are not physical threshold scales as well; these are bare parameters, the corresponding physical thresholds differ by the Z factors.

And yet, all these complications (the difference between the physical and bare thresholds and the higher orders in the β functions) conspire together and kill each other producing an effective picture of the one-loop evolution with the slopes jumping at fictitious thresholds. The actual evolution can be obtained from the fictitious straight lines of Fig. 1 by slightly distorting the scale in the horizontal direction. The distortion is logarithmic, i.e. rather weak (provided all couplings enter in the weak regime everywhere below M_0) and is scale (and pattern) dependent. In the $SU(3)$ and $SU(2)$ parts we pull the thresholds to the left, while in the $U(1)$ parts we pull them to the right, by different amounts. The distortions of the μ scale are determined by the Z factors which, although different from unity, are not strongly different. Under realistic conditions they may be 1.5 or 2 or so.

Thus, we witness another little miracle. Although the derivation presented above refers to a particular case, actually the message is general – (i) use the bare values of the threshold parameters abstracted from the bare Lagrangian God-given at some high (say, Planck) scale; (ii) forget about the multiloop effects, just run and match the gauge couplings according to the most naive one-loop formula (the so-called ϑ function approximation). Then, at the very end, when the evolution is finished, you will get the correct full results for the low-energy gauge constants.

In particular, let us draw attention to the fact that the additional matter produces no impact whatsoever in the difference of the couplings, Eq. (13). If we worked in terms of the physical thresholds, we would start from different physical masses for the three species of the additional matter. The Z factors will also be different. But the product of m 's times Z 's will be the same. That's how the additional matter contribution will disappear in the difference of the gauge couplings, although it will be much harder to see this disappearance keeping in mind that the Z factors are usually calculated approximately, not exactly.

This feature is general. Say, in a realistic $SU(5)$ model all $SU(5)$ multiplets provided by a common mass parameter are irrelevant for the calculation of the coupling differences.

What will happen if we introduce physical threshold scales – the “GUT” scales $\sqrt{3}\varphi^8/2$, instead of the bare scales? Then

$$\ln\left(\frac{\sqrt{3}\varphi_0^8}{2\varphi_0^3}\right) \longrightarrow \ln\left(\frac{\sqrt{3}\varphi^8}{2\varphi^3}\right)$$

plus logarithm of the Z factor describing the evolution of the ϕ field in the interval between the first and the second elephant threshold. If we transfer this logarithm

from the right-hand side to the left-hand side of Eq. (13), on the left-hand side we will get a specific combination which is actually nothing else than the Wilsonian couplings [15]. The issue of the thresholds will be discussed in more detail later on.

4 What if the evolution is not completed?

The example discussed above is very elegant but, unfortunately, is not very practical since usually the symmetry breaking pattern is such that not all gauge bosons acquire masses, so it is impossible to descend down, below the “freezing point”. If the freezing points existed, the scale where they (the freezing points) occur is established as a result of a natural flow, and is different in different subgroups. This variation of scales neutralizes the effect of the second and higher-order terms in the β function.

In the case there is no freezing point, or we measure the running couplings above the freezing points, we usually measure them at a scale μ which is one and the same for all subgroups. Then the impact of the second and higher-order terms in the β function is nonvanishing. To get a better idea of what is going on we will consider another pedagogical example, the $SU(3)$ model we dwell upon above, but this time it will be assumed that the symmetry breaking pattern is $SU(3) \rightarrow SU(2) \times U(1)$. Moreover, for the time being we will discard the additional matter, and add the superpotential term to the Φ superfield

$$\mathcal{W} = \frac{1}{4g_0^2} \left(m_0 \Phi^a \Phi^a + \lambda_0 d_{abc} \Phi^a \Phi^b \Phi^c \right), \quad (14)$$

where d_{abc} are the d symbols of $SU(3)$. The superpotential (14) destroys the valley, and fixes the vacuum value of the Φ field. One of the possible solutions is

$$\varphi_0^8 = \frac{2m_0}{\sqrt{3}\lambda_0};$$

this solution is characterized by the property of vanishing φ^3 which ensures that the $SU(2)$ subgroup remains unbroken. The bare mass squared of the fourth, fifth, sixth and seventh gauge bosons, the elephants, is

$$\left(M_{4-7}^{(0)} \right)^2 = (3|\varphi_0^8|^2)/4;$$

they eat up the corresponding Φ superfields. The first three gauge bosons remain massless. There are two thresholds – one associated with the elephants, another one with the fields $\Phi^{1,2,3}$. The latter are massive because of the superpotential (14) and, in particular, the mass term of the Φ superfield. The bare mass of $\Phi^{1,2,3}$ is

$$\left(M_{1-3}^{(0)} \right)^2 = |m_0|^2.$$

Let us assume that the elephant threshold is heavier than that of $\Phi^{1,2,3}$, which is natural if the coupling constant λ is weak. Then below the lowest threshold

associated with $\Phi^{1,2,3}$ we are left with $SU(2)$ supersymmetric gluodynamics, with no matter fields, plus the eighth photon and no charged matter. So, the eighth gauge coupling is frozen; the $SU(2)$ coupling, however, evolves till the very end – this subgroup, being unbroken, has no natural freezing point.

Thus, far above the thresholds we start from the unified gauge coupling α_0 . Far below the two thresholds we have the evolving coupling, $\alpha_{SU(2)}$, and the frozen coupling, $\alpha_{U(1)}$. Since the $SU(2)$ coupling evolves we have to distinguish between the Wilsonian constant – it will be marked by braces – and that appearing in the c -number functional Γ [15]. The Wilsonian constant $\{1/\alpha\}$ is renormalized only at one loop, while $1/\alpha$ is renormalized to all orders. The exact relation between $\{1/\alpha\}$ and $1/\alpha$ is explicitly known [15].

Let us choose the normalization point μ far below the lowest threshold, $\mu \ll M_{1-3}$. We start from the following one-loop formula for the Wilsonian couplings (which is perturbatively exact)

$$\left\{ \frac{2\pi}{\alpha_{SU(2)}(\mu)} \right\} = \frac{2\pi}{\alpha_0} - 6 \ln \frac{M_0}{\mu} - 3 \ln \frac{M_0}{(\sqrt{3}\varphi_0^8/2)} + 2 \ln \frac{M_0}{m_0} + \ln \frac{M_0}{(\sqrt{3}\varphi_0^8/2)}, \quad (15)$$

and

$$\frac{2\pi}{\alpha_{U(1)}} = \frac{2\pi}{\alpha_0} - 6 \ln \frac{M_0}{(\sqrt{3}\varphi_0^8/2)}. \quad (16)$$

The first line in Eq. (15) reflects the contribution of the gauge bosons, and the fact that the fourth to seventh become elephants and freeze out. The second line in Eq. (15) is due to the Φ fields. The first term in the second line comes from $\Phi^{1,2,3}$, the second from those Φ 's (Φ^4 to Φ^7) which are eaten up by the elephants. We know already that Eqs. (15) and (16) are valid to all orders.

The expressions presented above are not so easy to grasp; they may even generate a couple of perplexing questions. To get a transparent interpretation of this result two further steps are in order. First, we will pass from the Wilsonian coupling $\{1/\alpha_{SU(2)}(\mu)\}$ to the regular one, $1/\alpha_{SU(2)}(\mu)$. Second, it is convenient to consider the difference between $1/\alpha_{SU(2)}(\mu)$ and $1/\alpha_{U(1)}$, keeping in mind that the latter is frozen below the elephant threshold. Therefore, $\alpha_{U(1)}$ in the problem at hand is nothing else than α_{GUT} .

In pure $SU(2)$ supersymmetric gluodynamics the first transition is realized as follows [15]

$$\left\{ \frac{2\pi}{\alpha_{SU(2)}(\mu)} \right\} = \frac{2\pi}{\alpha_{SU(2)}(\mu)} + 2 \ln \frac{\alpha_{SU(2)}(\mu)}{\alpha_0}, \quad (17)$$

where the factor 2 in front of the logarithm is actually $T(G)$ for $G = SU(2)$. In addition, we have to introduce the Z factors describing the evolution of the Φ matter. Let $Z(M_0 \rightarrow M_{GUT})$ express the evolution from M_0 down to the physical GUT scale

$M_{4-7} \equiv M_{\text{GUT}}$, while $Z(M_0 \rightarrow M_{1-3})$ expresses the evolution from M_0 down to the physical m threshold (for Φ^1 to Φ^3). Then, with our definitions, $m_0 = Z(M_0 \rightarrow M_{1-3})m$ and $\sqrt{3}\varphi_0^8/2 = M_{\text{GUT}}[Z(M_0 \rightarrow M_{\text{GUT}})]^{-1/2}\alpha_0^{1/2}\alpha_{\text{GUT}}^{-1/2}$. Subtracting Eqs. (15) and (16) from each other and using the physical thresholds we readily get

$$\frac{2\pi}{\alpha_{\text{SU}(2)}(\mu)} = \frac{2\pi}{\alpha_{\text{GUT}}} - 6 \ln \frac{M_{\text{GUT}}}{\mu \left(\frac{\alpha_{\text{GUT}}}{\alpha_{\text{SU}(2)}(\mu)} \right)^{1/3}} + 2 \ln \frac{M_{\text{GUT}}}{mZ(M_{\text{GUT}} \rightarrow M_{1-3})}, \quad (18)$$

where the factor

$$Z(M_{\text{GUT}} \rightarrow M_{1-3}) = Z(M_0 \rightarrow M_{1-3})/Z(M_0 \rightarrow M_{\text{GUT}})$$

expresses the evolution from the elephant (GUT) threshold to the m threshold; thus, the product $mZ(M_{\text{GUT}} \rightarrow M_{1-3})$ is nothing else than the mass parameter M_{1-3} at the GUT scale (rather than the physical mass of Φ^{1-3}).

Equation (18) has various appealing features. In particular, the parameters M_0 and α_0 referring to the initial scale, lying far above the GUT scale, drop out, as they should.

Now, the last exercise aimed at making our toy model of grand unification a little bit more realistic. Let us add an extra matter superfield, say, in the fundamental representation of SU(3) (actually we will need two chiral superfields – one triplet and another antitriplet), with no mass term. What happens then?

A qualitative impact is quite obvious – the U(1) coupling constant now runs below the GUT threshold, and never freezes. Therefore, it is necessary to introduce now the Wilsonian coupling for U(1) as well. Quantitatively, an extra term appearing on the right-hand side of Eqs. (15) and (16) is

$$\ln \frac{M_0}{\mu},$$

one and the same in both expressions. Simultaneously, $2\pi/\alpha_{\text{U}(1)}$ in Eq. (16) should be substituted by $\{2\pi/\alpha_{\text{U}(1)}\}$. Transition from the Wilsonian couplings to the regular ones is now to be carried out in both subgroups, SU(2) and U(1). In the first case, instead of Eq. (17) we have

$$\left\{ \frac{2\pi}{\alpha_{\text{SU}(2)}(\mu)} \right\} = \frac{2\pi}{\alpha_{\text{SU}(2)}(\mu)} + 2 \ln \frac{\alpha_{\text{SU}(2)}(\mu)}{\alpha_0} + \ln Z(M_0 \rightarrow \mu; d.m.), \quad (19)$$

where d.m. stands for the doublet matter (s.m. below stands for singlet matter, i.e. the matter field from the original triplet which does not belong to the SU(2) doublet).

Furthermore, for the U(1) subgroup

$$\left\{ \frac{2\pi}{\alpha_{\text{U}(1)}(\mu)} \right\} = \frac{2\pi}{\alpha_{\text{U}(1)}(\mu)} + \frac{1}{3} \ln Z(M_0 \rightarrow \mu; d.m.) + \frac{2}{3} \ln Z(M_0 \rightarrow \mu; s.m.). \quad (20)$$

From M_0 down to M_{GUT} the Z factor is common, below the GUT scale it splits. The $\text{SU}(2)$ doublet interacts with both, the $\text{SU}(2)$ gauge bosons, and the eighth photon (see Eq. (11)). The $\text{SU}(2)$ singlet interacts only with the photon.

Assembling all pieces together we observe that after the additional matter is included, the following relation emerges

$$\begin{aligned} & \frac{2\pi}{\alpha_{\text{SU}(2)}(\mu)} - \frac{2\pi}{\alpha_{\text{U}(1)}(\mu)} = \\ & -6 \ln \frac{M_{\text{GUT}}}{\mu \left(\frac{\alpha_{\text{GUT}}}{\alpha_{\text{SU}(2)}(\mu)} \right)^{1/3}} + 2 \ln \frac{M_{\text{GUT}}}{mZ(M_{\text{GUT}} \rightarrow M_{1-3})} + \ln \frac{M_{\text{GUT}}}{\mu Z(M_{\text{GUT}} \rightarrow \mu; d.m.)} - \\ & \ln \frac{M_{\text{GUT}}}{\mu [Z(M_{\text{GUT}} \rightarrow \mu; d.m.)]^{1/3} [Z(M_{\text{GUT}} \rightarrow \mu; s.m.)]^{2/3}}, \end{aligned} \quad (21)$$

where each term in each line has a very clear physical interpretation. The three terms in the second line are due to the $\text{SU}(2)$ gauge bosons, the heavy mass threshold of the Φ fields; and the additional matter; the third line is the evolution of the $\text{U}(1)$ gauge coupling.

Our toy example is now generic enough in the sense that it includes all elements one may encounter in the realistic models. Therefore, at this stage we are able to formulate general conclusions. We start from the discussion of the mass thresholds.

5 Mass thresholds

The proper treatment of the heavy threshold effects in the evolution of the gauge couplings is crucial in establishing relations between the low-energy couplings following from the fact of unification at a higher scale. At the one-loop level there is no problem – the so called ϑ function (or run-and-match) approximation is applicable. According to this approximation, as μ evolves from M_{GUT} downwards, one just abruptly changes the first coefficient of the β function (i.e. the coefficient in front of the logarithm) at the physical value of the corresponding mass. Say, at the Φ threshold, above M_Φ one includes in the coefficient the Φ contribution while immediately below M_Φ one excludes the Φ fields altogether. At the one-loop level the ϑ function prescription is certainly correct.

The question arises beyond the leading log approximation where the above prescription is certainly incorrect. Following a tradition established in the pre-supersymmetric era the same ϑ function approximation is usually applied at two loops [21]. Even though numerically this may not be such a bad idea, it is interesting to ask what is the correct way of taking the heavy mass thresholds in the gauge couplings into account.

An idea which immediately comes to one's mind is smoothing the ϑ functions in some way [22]. The smoothing procedure is rather ambiguous, it depends on

conventions and, generally speaking, has no direct physical meaning. This happens because the gauge couplings by themselves are not physical observables. The definition we follow – through the coefficient in front of W^2 in the effective action – is directly related to observables (with the power accuracy) only provided we are far above the threshold or far below it. Fortunately, with respect to the heavy thresholds, we are always far below. If we calculate at low energies the effect due to the heavy fields (which are integrated out since they can appear only in virtual loops) in the *full* theory it is all we need. And, as was shown above, that's exactly what we do.

Let us examine Eq. (21) (the same conclusions follow from the analysis of Sect. 3). The heavy threshold effect in this expression is represented by the second term in the second line,

$$2 \ln \frac{M_{\text{GUT}}}{mZ(M_{\text{GUT}} \rightarrow M_{1-3})}.$$

If the physical mass of the fields Φ^{1-3} coincides with M_{GUT} (i.e. with the elephant masses), then the argument of the logarithm is unity – there is no heavy threshold correction at all. This is quite natural and uninteresting. What is more interesting is the case when the physical mass of the fields Φ^{1-3} is less than M_{GUT} . Then the threshold correction is non-vanishing. In the ϑ function approximation the threshold correction will reduce to

$$2 \ln \frac{M_{\text{GUT}}}{m}$$

where m is the physical mass of the fields Φ^{1-3} (the same as M_{1-3}).

Now, the correct answer, taking into account all orders, replaces the physical mass m by $mZ(M_{\text{GUT}} \rightarrow M_{1-3})$, or the corresponding *mass parameter at the GUT scale*. If our research starts from model-building and goes in the direction of phenomenology, the short distance (GUT) value of the mass parameter is primary, and no further efforts are required. Case closed. If, however, we start from phenomenology, and operate with the physical masses of the particles, then to properly include the heavy threshold effects beyond one loop (i.e. beyond the ϑ function approximation) one has to calculate the corresponding Z factors. Unfortunately, the Z factors are not calculable to all orders. Note, however, that in the two-loop analysis of the gauge couplings the Z factors appear only at one-loop (i.e. in the leading log approximation). In other words, we reduce the level of complexity by one order. In particular, if there are heavy threshold corrections in the Z factors themselves (they occur more rarely than in the gauge couplings, if at all) it is legitimate to account for them in the ϑ function approximation. This will provide us with the two-loop evolution of the gauge couplings properly accounting for the heavy thresholds.

6 General lessons and master formula

In order to analyze the evolution of the gauge couplings from M_{GUT} down to some low scale, first one starts from the standard one loop expressions written in the most

primitive “run and match” approximation, using the physical values of the mass parameters in the logarithms. By the mass parameters I mean the threshold masses, M_{GUT} , and the normalization point μ . Then, in the logarithms corresponding to the matter field contribution, substitute the mass parameter by the mass parameter multiplied by the corresponding Z factor. In the logarithms corresponding to the gauge field contribution, substitute the mass parameter by the mass parameter multiplied by the corresponding $Z^{1/3}$ factor. For the gauge fields the Z factor is $1/\alpha$.

The above procedure fully specifies the evolution process. It is perturbatively *exact to all orders*, and takes into account the *heavy mass thresholds exactly* provided that supersymmetry holds, i.e. we stay above the scale of the supersymmetry breaking.

(In practical calculations one usually limits oneself to two-loop accuracy. Even for that limited purpose our master formula is extremely useful. First, it spares one the necessity of tabulating the two-loop coefficients of the β function, a rather cumbersome task by itself. Only one loop coefficients enter the game, and all one loop coefficients are simple numbers having a geometric, and group-theoretic, meaning. Second, it provides with a consistent and exhaustive treatment of the mass thresholds, as was mentioned above. Third, numerical integration of the renormalization group equations becomes unnecessary. The final result is expressed in terms of the Z factors. In the two-loop analysis the latter enter only at one loop level, or, more exactly, in the leading log approximation, which is equivalent to one loop.)

In the text book case of the SU(5) grand unification [23] the master formula is

$$\begin{aligned} \frac{2\pi}{\alpha_3(\mu)} = \frac{2\pi}{\alpha_{\text{GUT}}} - 9 \ln \frac{M_{\text{GUT}}}{\mu \left(\frac{\alpha_{\text{GUT}}}{\alpha_3(\mu)} \right)^{1/3}} + \sum_{\text{gen}} \left[\ln \frac{M_{\text{GUT}}}{\mu Z_{qL}} + \frac{1}{2} \ln \frac{M_{\text{GUT}}}{\mu Z_{UR}} + \frac{1}{2} \ln \frac{M_{\text{GUT}}}{\mu Z_{DR}} \right] + \\ + 3 \ln \frac{M_{\text{GUT}}}{M_{\Phi}} + \ln \frac{M_{\text{GUT}}}{M_{H_{u,d}}^{(3)}}, \end{aligned} \quad (22)$$

$$\begin{aligned} \frac{2\pi}{\alpha_2(\mu)} = \frac{2\pi}{\alpha_{\text{GUT}}} - 6 \ln \frac{M_{\text{GUT}}}{\mu \left(\frac{\alpha_{\text{GUT}}}{\alpha_2(\mu)} \right)^{1/3}} + \sum_{\text{gen}} \left[\frac{3}{2} \ln \frac{M_{\text{GUT}}}{\mu Z_{qL}} + \frac{1}{2} \ln \frac{M_{\text{GUT}}}{\mu Z_{\ell L}} \right] + \\ + 2 \ln \frac{M_{\text{GUT}}}{M_{\Phi}} + \ln \frac{M_{\text{GUT}}}{M_{H_{u,d}}^{(2)}}, \end{aligned} \quad (23)$$

$$\begin{aligned} \frac{2\pi}{\alpha_1(\mu)} = \frac{2\pi}{\alpha_{\text{GUT}}} + \\ + \frac{3}{10} \left[\sum_{\text{gen}} \left(\ln \frac{M_{\text{GUT}}}{\mu Z_{\ell L}} + 2 \ln \frac{M_{\text{GUT}}}{\mu Z_{eR}} + \frac{1}{3} \ln \frac{M_{\text{GUT}}}{\mu Z_{qL}} + \frac{8}{3} \ln \frac{M_{\text{GUT}}}{\mu Z_{UR}} + \frac{2}{3} \ln \frac{M_{\text{GUT}}}{\mu Z_{DR}} \right) + \right. \end{aligned}$$

$$2 \ln \frac{M_{\text{GUT}}}{M_{H_{u,d}}^{(2)}} + \frac{4}{3} \ln \frac{M_{\text{GUT}}}{M_{H_{u,d}}^{(3)}} \Bigg] . \quad (24)$$

Here I made various simplifying assumptions (which, in principle, can be easily lifted). First, it is assumed, of course, that μ is larger than the superpartners' masses – our master formula is valid only as long as supersymmetry holds. The masses of the two Higgs doublets of the model at hand, $M_{H_{u,d}}^{(2)}$ are also assumed to lie above the supersymmetry breaking scale, and above μ . Let us stress that $M_{H_{u,d}}^{(2)}$ in the expressions above is *not* the physical mass, but, rather the corresponding mass term at the GUT scale. At the same time the SU(2) breaking scale is taken to be below μ – the $SU(3) \times SU(2) \times U(1)$ symmetry is manifest in Eqs. (22) – (24). Moreover, the Yukawa interactions are neglected so that all three matter generations have identical Z factors. This means that in the approximation accepted there are only five different matter Z factors – Z_{qL} corresponding to the left-handed quark doublet, Z_{UR} corresponding to the right-handed up-quark, Z_{DR} corresponding to the right-handed down-quark, $Z_{\ell L}$ corresponding to the left-handed lepton doublet, and, finally, Z_{eR} corresponding to the right-handed electron (muon, τ). $M_{H_{u,d}}^{(3)}$ stands for the mass term of the triplet superheavy Higgs particles (from $\mathbf{5}$ and $\bar{\mathbf{5}}$), as it is seen at the GUT scale. Finally, M_Φ is the mass term of the chiral superfields from the 24-plet at the GUT scale. The last line in each of the expressions above represents the GUT scale threshold terms, plus the doublet Higgs contribution in Eqs. (23), (24).

All the simplifying assumptions listed above can be lifted in an obvious way.

7 Z factors

The master formula above expresses the gauge couplings, to all orders, in terms of the Z factors. Unfortunately, the latter are not calculable to all orders. The Z factors appear as the coefficients in front of the D terms, and the theorems of the holomorphic dependence on the moduli and the (inverse) gauge couplings are not valid [24]. Without the power of holomorphicity one can not establish the all-order results for the Z factors and has to resort to the old-fashioned order-by-order perturbative calculations. The anomalous dimensions of the matter fields were calculated up to three loops [25]. The higher-order terms in γ 's are definition-dependent, however, starting from the second loop. The work of bringing the results in line with the definition where the NSVZ β function is valid is under way now [26]. After this work is done one will be able to calculate the gauge coupling evolution up to fourth order.

As was mentioned more than once, for the two-loop analysis of the gauge couplings, it is sufficient to have the Z factors in the leading log approximation, which is unambiguous and scheme-independent. Actually, the one loop expressions can be obtained with *no calculations*, from general arguments alone.

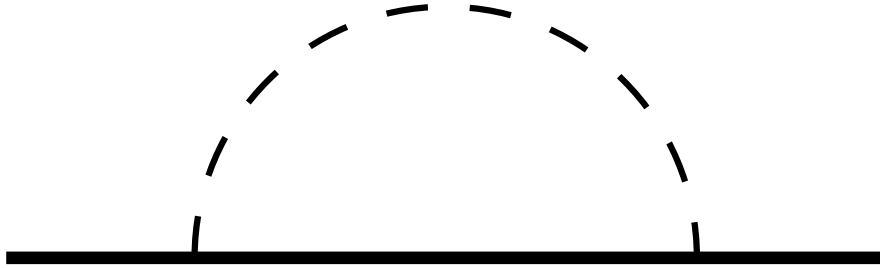


Figure 2: One-loop graph describing the matter field renormalization due to the exchange of the gauge field.

The relevant one-loop diagram is presented on Fig. 2. Let us assume at first that we have only one chiral matter superfield, in the adjoint representation, with no self-interaction. Then the theory has $N = 2$ supersymmetry, and the β function is well-known to be one-loop. From Eq. (10) we then immediately read off that

$$\gamma = -N \frac{\alpha}{\pi}.$$

From the structure of the graph depicted on Fig. 2 it is clear that the factor N above is nothing else than $C_2(\text{adj})$ where C_2 is the quadratic Casimir operator,

$$(T^a T^a)_{\text{rep } R} = C_2(R) \mathbf{1},$$

related to the Dynkin index in a standard way,

$$C_2(R) = T(R) \frac{\dim(\text{adj})}{\dim(R)}.$$

At the one loop order the diagram of Fig. 2 is the only relevant graph. It is obvious then that in the general case of the matter field in the arbitrary representation of the gauge group, $C_2(\text{adj})$ is merely substituted by $C_2(R)$, irrespective of how many matter fields we have, whether or not there are Yukawa interactions (self-interactions), and so on. We conclude that the one-loop gauge contribution to Z is

$$Z = 1 - C_2(R) \frac{\alpha_0}{\pi} \ln \frac{M_0}{\mu}. \quad (25)$$

It is clear that the gauge-interaction contribution is negative, tending to make Z 's less than unity.

The renormalization group improvement of expression (25) (summation of the leading logarithms) is trivial,

$$Z \rightarrow \left(\frac{\alpha_0}{\alpha} \right)^{2C_2/b_0}, \quad (26)$$

where

$$b_0 = 3T(G) - \sum_i T(R_i)$$

is the first coefficient in the β function.

Similar “back-of-the-envelope” arguments yield the (super)Yukawa vertices’ contribution to Z ’s, which, on the contrary, tends to make the Z factors larger than unity. Indeed, let us start again from the case of adjoint matter, this time assuming that there are three matter fields, Φ_1^a , Φ_2^a , and Φ_3^a . If the Yukawa coupling

$$\Delta\mathcal{W} = \frac{1}{g_0^2} f^{abc} \Phi_1^a \Phi_2^b \Phi_3^c \quad (27)$$

is added to the superpotential the theory becomes $N = 4$ supersymmetric, the gauge coupling does not run, and the anomalous dimensions of all Φ fields vanish. This means that the renormalization of, say, the Φ_1 field due to the gauge boson (Fig. 2) is canceled by a similar diagram with the gauge boson line replaced by that of the Φ fields, and the gauge vertices replaced by the (super)Yukawa vertices. We immediately conclude then that

$$Z_{\text{Yukawa}} = 1 + N \frac{\alpha_0}{\pi} \ln \frac{M_0}{\mu}. \quad (28)$$

Getting the Yukawa contributions from Eq. (28) for other matter fields and different structure of the Yukawa vertices is a mere substitution of obvious color factors.

It is convenient to collect here the leading-logarithm expressions for the five Z factors entering our master formula in the $\text{SU}(5)$ case,

$$\begin{aligned} Z_{qL} &= Z_{\text{SU}(3)} \times Z_{\text{SU}(2)} \times Z_1^{1/9}, \\ Z_{UR} &= Z_{\text{SU}(3)} \times Z_1^{16/9}, \\ Z_{DR} &= Z_{\text{SU}(3)} \times Z_1^{4/9}, \\ Z_{\ell L} &= Z_{\text{SU}(2)} \times Z_1, \end{aligned}$$

and, finally,

$$Z_{eR} = Z_1^4, \quad (29)$$

where

$$Z_{\text{SU}(3)} = \left[\frac{\alpha_{\text{GUT}}}{\alpha_3(\mu)} \right]^{8/9}, \quad Z_{\text{SU}(2)} = \left[\frac{\alpha_{\text{GUT}}}{\alpha_2(\mu)} \right]^{-3/2}, \quad Z_1 = \left[\frac{\alpha_{\text{GUT}}}{\alpha_1(\mu)} \right]^{-1/22}, \quad (30)$$

and the Yukawa interactions are neglected, for simplicity. The Yukawa interactions might be important for the t quark, but even in this case, numerically, their effect is very modest. Moreover, it is assumed that the heavy thresholds coincide with the GUT scale, while the doublet Higgs particles are lighter than μ . It is quite clear how to amend this expression if μ is smaller than the doublet Higgs masses.

8 Numerical analysis: bird's eye view

Now, we will discuss the numerical situation, in gross features, leaving aside fine details. Our task is merely to get a general idea. To this end I will neglect the low-energy thresholds due to supersymmetry breaking, assuming that all superpartners have masses in the ballpark of 100 to 200 GeV. Making some of them significantly heavier (say, ~ 1 TeV) may change our conclusions. This is a different topic, however, which I will not touch upon, referring the reader to the very rich literature devoted to the issue of low-energy thresholds. Two other possibilities – new physics in the middle of the great desert, and Planck physics corrections at the GUT scale – will not be considered as well. The first topic was covered in the talks of B. Brachmachari, E. Keith and some others, while the second topic was extensively reviewed by Prof. P. Nath.

To begin with, let us truncate the expressions (22) – (24) at one loop and disregard the heavy thresholds (i.e. assume that the masses of the Higgses from the 24-plet and triplets from the quintets coincide with M_{GUT}). Then for the low-energy value of α_3 we get

$$\frac{2\pi}{\alpha_3} = \frac{12}{7} \frac{2\pi}{\alpha_2} - \frac{5}{7} \frac{2\pi}{\alpha_1}, \quad (31)$$

where by low energy I mean that the corresponding normalization point is somewhere in the ballpark of 100 to 200 GeV, see above. The value of $2\pi/\alpha_2$ is close to 186 and that of $2\pi/\alpha_1$ is close to 371. Where exactly the normalization point lies affects the numbers above at the level \sim one unit; a similar uncertainty is associated with the low-energy thresholds (provided that the sparticle masses are ~ 100 GeV). The effects I am hunting for now constitute seven units – this is the impact of the difference between $\alpha_s(M_Z) = 0.11$ and $\alpha_s(M_Z) = 0.125$.

Using Eq. (31) we conclude that $2\pi/\alpha_3 \approx 54$ (and $2\pi/\alpha_{\text{GUT}} \approx 153$). If the second loop shifted this number up by three units, to 57, I would be very happy. The above value then would nicely match what is expected from the low-energy QCD phenomenology, $\alpha_s(M_Z) \approx 0.11$ [3]. Unfortunately, the higher-order corrections work just in the opposite direction shifting $2\pi/\alpha_3$ down by three units.

By inspecting Eqs. (22) – (24) and (29) we immediately observe that, by far, the most significant numerical effect of higher loops comes from the factors

$$\left(\frac{\alpha_{\text{GUT}}}{\alpha_3(\mu)}\right)^{1/3} \quad \text{and} \quad \left(\frac{\alpha_{\text{GUT}}}{\alpha_2(\mu)}\right)^{1/3}$$

in the logarithms in Eqs. (22), (23) and from $Z_{\text{SU}(3)}$. The factors Z_1 and $Z_{\text{SU}(2)}$ in the standard version of the SU(5) grand unification are too close to unity to be of importance in my back-of-the-envelope analysis – they are buried in the noise of other insignificant corrections. Keeping only the important factors we get, instead of Eq. (31),

$$\frac{2\pi}{\alpha_3} = \frac{12}{7} \frac{2\pi}{\alpha_2} - \frac{5}{7} \frac{2\pi}{\alpha_1} +$$

$$3 \ln \left(\frac{\alpha_{\text{GUT}}}{\alpha_3(\mu)} \right) - \frac{24}{7} \ln \left(\frac{\alpha_{\text{GUT}}}{\alpha_2(\mu)} \right) - \frac{9}{14} \ln Z_{\text{SU}(3)}. \quad (32)$$

In the second line we can use the values of the constants in the one-loop approximation; moreover, $Z_{\text{SU}(3)}$ is given by Eq. (30). The most significant contribution, -3 , comes from the first term in the second line. The second and the third terms are close to 0.6 and nearly compensate each other. As a result, the second loop shifts the prediction for $2\pi/\alpha_3$ away from the right value, from 54 to 51 (i.e. $\alpha_s(M_Z) \approx 0.123$).

Since we are so remarkably close to the desired number it is quite natural to think that the residual discrepancy is due to some corrections that are unaccounted for so far. The idea which immediately comes to one's mind is including the Yukawa interactions in order to change the values of the Z factors. We – Ian Kogan and myself – played with this idea for some time. To achieve success in this way one has to make an appropriate Z factor of order 100 or larger ($\ln Z \sim 5$). It is quite obvious that in the minimal model even the strongest Yukawa interaction, that of the t quark, is far too weak to ensure such a large value of Z . Only if some Yukawa constant approaches the Landau pole can we expect to get Z that large.

Another way out may be associated with the heavy particles' thresholds. Generally speaking, the masses of the Higgs particles from the 24-plet and the triplets from the quintets need not exactly coincide with M_{GUT} , see the second lines in Eqs. (22) – (24). Again, it is clear that to shift the prediction for $2\pi/\alpha$ by several units the logarithm of the mass ratio has to be several units itself. In other words, the superheavy Higgs particles have to be hundred times lighter than M_{GUT} . In this case there is the menace of a fast proton decay due to dimension five operators [27]. In more details the scenario was investigated in Refs. [28, 29, 30]. It was shown that in the $\text{SU}(5)$ grand unification this solution does not work, unfortunately. The proton decays too fast [28, 29]. In the $\text{SO}(10)$ grand unification, however, the superheavy Higgs thresholds do bring the value of $\alpha_s(M_Z)$ down to 0.11 without making the proton lifetime unacceptably short [30]. Moreover, the prediction for the proton lifetime returns to the range accessible experimentally, which makes the whole story exciting.

9 Conclusions

- If the primary “high energy” gauge group G is spontaneously broken, $G \rightarrow G_1 \times G_2 \times \dots$, then the low-energy couplings $1/\alpha_1, 1/\alpha_2, \dots$ are given by a master formula *to all orders*, including mass thresholds.
- If supersymmetric mass parameters are known at the GUT scale (rather than the physical mass parameters) then full inclusion of the threshold effects trivializes.
- If the evolution is complete (i.e. all G_i 's are $U(1)$'s, and the low-energy couplings are frozen, then the all-order result for $1/\alpha_i$ looks as if it was one-loop, with *faked* (bare) thresholds.

- The center of gravity of the multiloop analyses is shifted from the gauge β functions to the Z factors of the matter fields. The complexity (number of loops) is reduced by one level. In the minimal SU(5) grand unification the dominant two-loop correction comes from the one-loop Z factor of the gluon (gluino) field. It is of the right magnitude, but, unfortunately, its sign is “unfavorable”, so that the value of $\alpha_s(M_Z)$ becomes unacceptably high.

10 Acknowledgments

I would like to thank I. Kogan, L. Roszkowski, and A. Vainshtein for stimulating discussions. Assistance of B. Chibisov with plots is gratefully acknowledged. This work was supported in part by DOE under the grant number DE-FG02-94ER40823.

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